

PARTITIONING BASES OF TOPOLOGICAL SPACES

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ABSTRACT. We investigate whether an arbitrary base for a dense-in-itself topological space can be partitioned into two bases. We prove that every base for a T_3 Lindelöf topology can be partitioned into two bases while there exists a consistent example of a first countable, 0-dimensional, Hausdorff space of size 2^ω and weight ω_1 which admits a point countable base without a partition to two bases.

1. INTRODUCTION

At the Trends in Set Theory conference in Warsaw, Barnabás Farkas¹ raised the natural question whether one can partition any given base for a topological space into two bases; we will call this property being *base resolvable*. Note that every space with an isolated point is not base resolvable; hence, from now on by *space* we mean a *dense-in-itself topological space*. The aim of this paper is to present two streams of results: in the first part of the article, we will show that certain natural classes of spaces are base resolvable. In the second part, we present a method to construct non base resolvable spaces.

The paper is structured as follows: in Section 2, we will start with general observations about bases and we prove that metric spaces and left- or right-separated spaces are base resolvable. This section also serves as an introduction to the methods that will be applied in Section 3 where we prove one of our main results in Theorem 3.6: every T_3 (locally) Lindelöf space is base resolvable.

In Section 4, we investigate base resolvability from a purely combinatorial viewpoint which leads to further results: every hereditarily Lindelöf space (without any separation axioms) is base resolvable and any base for a T_1 topology which is closed to finite unions can be partitioned into two bases, see Theorem 4.6 and 4.7 respectively.

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Next in Theorem 5.6, we prove that every base \mathbb{B} for a space X (resolvable or not) contains a large *negligible* portion, i.e. there is $\mathcal{U} \in [\mathbb{B}]^{|\mathbb{B}|}$ such that $\mathbb{B} \setminus \mathcal{U}$ is still a base for X .

The second part of the paper starts with Section 6; here, we isolate a partition property, denoted by $\mathbb{P} \rightarrow (I_\omega)_2^1$, of the partial order $\mathbb{P} = (\mathbb{B}, \supseteq)$ associated to a base \mathbb{B} which is closely related to base resolvability. We will construct a partial order \mathbb{P} with this property in Theorem 6.5 and deduce the existence of a T_0 non base resolvable topology (in ZFC) in Corollary 6.13.

Next, in Section 7 we present a ccc forcing (of size ω_1) which introduces a first countable, 0-dimensional, Hausdorff space X of size 2^ω and weight ω_1 such that X is not base resolvable. The main ideas of the construction already appear in Section 6 however the details here are much more subtle and the proofs are more technical.

The paper finishes with a list of open problems in Section 8. We remark that Section 7 was prepared by the second author and the rest of the paper is the work of the first author.

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2. GENERAL RESULTS

In this section, we prove some basic results concerning partitions of bases; these proofs will introduce us to the more involved techniques of the upcoming sections.

Definition 2.1. *A base \mathbb{B} for a space X is **resolvable** iff it can be decomposed into two bases. A space X is **base resolvable** if every base of X is resolvable.*

Recall that by *space* we will mean a dense-in-itself topological space throughout the paper.

Partitioning sets with additional structure is a highly investigated theme in mathematics; let us cite a classical result of A. H. Stone which is relevant to our case:

Theorem 2.2 (A. H. Stone, [2]). *Every partially ordered set (\mathbb{P}, \leq) without maximal elements can be partitioned into two cofinal subsets.*

Proposition 2.3. *(1) Every base can be partitioned to a cover and a base.*

- (2) Every π -base can be partitioned to two π -bases.
- (3) Every neighborhood base can be partitioned to two neighborhood bases.

Proof. To prove (1), note that every cover contains a well founded (with respect to \subset) subcover. Also, well founded families of open sets cannot form neighborhood bases in dense-in-itself spaces; thus, if \mathcal{U} is a well founded cover of X and \mathbb{B} is a base then $\mathbb{B} \setminus \mathcal{U}$ is still a base of X .

Note that (2) and (3) follows from Theorem 2.2. \square

Now we prove our first general result.

Proposition 2.4. *Every space with a σ -disjoint base is base resolvable; in particular, every metrizable space is base resolvable.*

Proof. Fix a space X with a base $\cup \mathbb{E}_n$ where each \mathbb{E}_n is a disjoint family; fix an arbitrary base \mathbb{B} as well which we aim to partition.

By induction on $n \in \omega$, construct $\mathbb{B}_{i,n} \subseteq \mathbb{B}$ for $i < 2$ such that

- (1) $\mathbb{B}_{i,n}$ is well founded for $i < 2$, $n \in \omega$,
- (2) $\mathbb{B}_{i,n} \cap \mathbb{B}_{j,m} = \emptyset$ if $i, j < 2$, $n, m \in \omega$ and $(i, n) \neq (j, m)$,
- (3) for every $V \in \mathbb{E}_n$ and $i < 2$ there is $\mathcal{U} \subseteq \mathbb{B}_{i,n}$ such that $\cup \mathcal{U} = V$.

Note that property (1) assures that $\mathbb{B} \setminus \cup \{\mathbb{B}_{i,k} : i < 2, k < n\}$ is still a base of X for each $n < \omega$ thus the induction can be carried out. Let $\mathbb{B}_i = \cup \{\mathbb{B}_{i,n} : n \in \omega\}$ for $i < 2$; it is easy to see that these disjoint families will form a base by property (3). \square

Note that every σ -disjoint base is point countable, however our example of an irresolvable base constructed in Section 7 is point countable.

A somewhat similar technique, which will be used later as well, gives the following result:

Proposition 2.5. *Suppose that a regular space X satisfies $L(X) < \kappa = w(X) = \min\{\chi(X, x) : x \in X\}$. Then X is base resolvable.*

Proof. Fix a base \mathbb{B} for X and an enumeration $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ of all pairs of elements $U, V \in \mathbb{B}$ such that $\overline{U} \subseteq V$; without loss of generality, we can suppose that \mathbb{B} has size κ .

By induction on $\alpha < \kappa$ construct $\mathbb{B}_{0,\alpha}, \mathbb{B}_{1,\alpha} \subseteq \mathbb{B}$ such that

- (1) $\mathbb{B}_{0,\alpha} \cap \mathbb{B}_{1,\alpha} = \emptyset$ and $\mathbb{B}_{i,\alpha} \subseteq \mathbb{B}_{i,\beta}$ for $\alpha < \beta < \kappa$ and $i < 2$,
- (2) there is $\mathcal{U} \subseteq \mathbb{B}_{i,\alpha}$ such that $\overline{U_\alpha} \subseteq \cup \mathcal{U} \subseteq V_\alpha$ for every $i < 2$,
- (3) $|\mathbb{B}_{i,\alpha}| \leq L(X) \cdot |\alpha|$ for $i < 2$.

Note that our assumptions on the space and the inductive hypothesis (3) implies that

$$\mathbb{B} \setminus \bigcup \{\mathbb{B}_{i,\beta} : \beta < \alpha, i < 2\}$$

is still a base for X for every $\alpha < \kappa$. It follows that the induction can be carried out and the disjoint families $\mathbb{B}_i = \cup\{\mathbb{B}_{i,\alpha} : \alpha < \kappa\}$ form a base for X by (2); thus X is base resolvable. \square

We end this section by giving further classes of spaces which are base resolvable.

Observation 2.6. *Every right or left separated space is base resolvable. Furthermore, the Sorgenfrey line or the Double Arrow space is base resolvable.*

Proof. Recall that every neighborhood base can be partitioned into two neighborhood bases by Proposition 2.3. Thus, if \mathbb{B} is a base of X and there is a map $f : \mathbb{B} \rightarrow X$ such that $f^{-1}(x)$ is a base at x for any $x \in X$ then by partitioning $f^{-1}(x)$ for each $x \in X$ into two neighborhood bases of x we get a partition of \mathbb{B} into two bases of X . Now, it is straightforward to finish the proof. \square

3. LINDELÖF SPACES ARE BASE RESOLVABLE

Our aim in this section is to prove that T_3 Lindelöf spaces are base resolvable; we start with a definition and some observations while the most important part of the work is done in the proof of Lemma 3.3.

Definition 3.1. *Let \mathcal{A}, \mathcal{B} families of open sets in a space X . We say that \mathcal{A} **weakly fills** \mathcal{B} iff for every $U, V \in \mathcal{B}$ such that $\overline{U} \subset V$ there is $\mathcal{W} \subseteq \mathcal{A}$ such that*

$$\overline{U} \subseteq \cup \mathcal{W} \subset V.$$

*\mathcal{A}, \mathcal{B} is called a **weakly good pair** iff \mathcal{A}, \mathcal{B} are disjoint, \mathcal{A} weakly fills \mathcal{B} and \mathcal{B} weakly fills \mathcal{A} .*

We remark that in the next section we introduce stronger notions called *filling* and *good pairs*. The following observations summarize the importance of weakly good pairs:

Observation 3.2. *Suppose that X is a regular space.*

- (1) *If $(\mathcal{A}, \mathcal{B})$ is a weakly good pair in X then \mathcal{A} contains a neighborhood base at x iff \mathcal{B} contains a neighborhood base at x , for any $x \in X$.*
- (2) *If a family of open sets \mathcal{A} weakly fills a base \mathbb{B} of X then \mathcal{A} is a base as well.*
- (3) *If $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ and $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ are increasing and $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ is a weakly good pair in X then $(\cup_{\alpha < \kappa} \mathcal{A}_\alpha, \cup_{\alpha < \kappa} \mathcal{B}_\alpha)$ is a weakly good pair as well.*

We encourage the reader to compare these observations with the proof of Proposition 2.5.

We say that the weakly good pair $(\mathcal{A}', \mathcal{B}')$ **extends** the weakly good pair $(\mathcal{A}, \mathcal{B})$ iff $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subseteq \mathcal{B}'$. A family of weakly good pairs $\{(\mathcal{A}_\xi, \mathcal{B}_\xi) : \xi < \Theta\}$ is **pairwise disjoint** iff $\mathcal{A}_\xi \cap \mathcal{B}_\zeta = \emptyset$ for each $\xi, \zeta < \Theta$.

Next, we prove that weakly good pairs can be nicely extended in Lindelöf spaces.

Lemma 3.3. *Suppose that X is a T_3 Lindelöf space with a base \mathbb{B} . Given a weakly good pair $(\mathcal{A}, \mathcal{B})$ from elements of \mathbb{B} and a single pair of open sets $\{U, V\}$ such that $\overline{U} \subset V$ there is a weakly good pair $(\mathcal{A}', \mathcal{B}')$ formed by elements of \mathbb{B} extending $(\mathcal{A}, \mathcal{B})$ such that both \mathcal{A}' and \mathcal{B}' weakly fills $\{U, V\}$.*

Proof. We will show this essentially by induction on the size of \mathcal{A} and \mathcal{B} however we need to prove something significantly stronger (and more technical) than the statement of the lemma itself.

Let Δ_κ stand for the following statement: for each pairwise disjoint family of weakly good pairs $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$, each a subfamily from \mathbb{B} , such that $|\mathcal{A}_i|, |\mathcal{B}_i| \leq \kappa$ and arbitrary open family \mathcal{E} of size at most κ there is a weakly good pair $(\mathcal{A}, \mathcal{B})$ from \mathbb{B} of size at most κ such that

- (1) $\cup_{i < n} \mathcal{A}_i \subset \mathcal{A}$ and $\cup_{i < n} \mathcal{B}_i \subset \mathcal{B}$,
- (2) \mathcal{A} and \mathcal{B} weakly fills \mathcal{E} ,
- (3) $\{(\mathcal{A}, \mathcal{B}), (\mathcal{C}_j, \mathcal{D}_j) : j < k\}$ is still pairwise disjoint.

We prove that Δ_κ holds for every infinite κ by induction on κ .

Claim 3.4. Δ_ω holds.

Proof. Fix $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ and \mathcal{E} as above. By induction on $m \in \omega$ we build increasing $\{\mathcal{A}^m : m \in \omega\}$ and $\{\mathcal{B}^m : m \in \omega\}$ such that $\mathcal{A}^m, \mathcal{B}^m$ are disjoint subfamilies of \mathbb{B} and

- (1) $\mathcal{A}^0 = \cup_{i < n} \mathcal{A}_i, \mathcal{B}^0 = \cup_{i < n} \mathcal{B}_i$,
- (2) $\mathcal{A}^{m+1} \setminus \mathcal{A}^m$ and $\mathcal{B}^{m+1} \setminus \mathcal{B}^m$ are countable well-founded families for each $m \in \omega$,
- (3) $\mathcal{A}^m \cap \mathcal{B}_i = \emptyset, \mathcal{A}^m \cap \mathcal{D}_j = \emptyset$ and $\mathcal{B}^m \cap \mathcal{A}_i = \emptyset, \mathcal{B}^m \cap \mathcal{C}_j = \emptyset$ for $i < n, j < k, m < \omega$.

Furthermore, we will make sure that $\mathcal{A} = \cup_{m \in \omega} \mathcal{A}^m$ and $\mathcal{B} = \cup_{m \in \omega} \mathcal{B}^m$ forms a weakly good pair and they both weakly fill \mathcal{E} . Therefore, we partition ω into infinite sets $\omega = \cup \{D_m : m \in \omega\}$ and at each step we define a surjective map $f_m : D_m \setminus (m+1) \rightarrow \{(U, V) \in (\mathcal{A}^m \cup \mathcal{B}^m \cup \mathcal{E})^2 :$

$\overline{U} \subset V$; if $m \in D_l \setminus (l+1)$ and $f_l(m) = (U, V)$ then at step m we extend so that \mathcal{A}^m and \mathcal{B}^m weakly fills $\{U, V\}$.

Now our goal is reduced to construct \mathcal{A}^{m+1} and \mathcal{B}^{m+1} from \mathcal{A}^m and \mathcal{B}^m such that they satisfy (2)-(3) above while they both weakly fill a given $\{U, V\}$. We construct \mathcal{A}^{m+1} , the proof for \mathcal{B}^{m+1} is analogous. Define

$$\begin{aligned} F_i &= \{x \in X : \mathcal{A}_i \text{ contains a neighborhood base at } x\} \\ &= \{x \in X : \mathcal{B}_i \text{ contains a neighborhood base at } x\} \end{aligned}$$

and

$$\begin{aligned} G_j &= \{x \in X : \mathcal{C}_j \text{ contains a neighborhood base at } x\} \\ &= \{x \in X : \mathcal{D}_j \text{ contains a neighborhood base at } x\}. \end{aligned}$$

For every $i < 2$ and $x \in F_i \cap \overline{U}$ pick $U_{x,i} \in \mathcal{A}_i$ such that $x \in U_{x,i} \subset V$; let $\mathcal{U} = \{U_{x,i} : i < 2, x \in F_i \cap \overline{U}\}$. For $j < k$ and $x \in G_j \cap \overline{U}$ pick $V_{x,j} \in \mathcal{C}_j$ such that $x \in V_{x,j} \subset V$; let $\mathcal{V} = \{V_{x,j} : j < k, x \in G_j \cap \overline{U}\}$. Now note that for every $x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})$ there is a neighborhood base for x in $\mathbb{B} \setminus \bigcup_{i < 2, j < k} (\mathcal{B}_i \cup \mathcal{D}_j)$; hence for every $x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})$ we can pick $W_x \in \mathbb{B} \setminus \bigcup_{i < 2, j < k} (\mathcal{B}_i \cup \mathcal{D}_j)$ such that $x \in W_x \subset V$; let $\mathcal{W} = \{W_x : x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})\}$. Select a countable well-founded subcover $\mathcal{Q} \subset \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$ of \overline{U} and define $\mathcal{A}^{m+1} = \mathcal{A}^m \cup \mathcal{Q}$. \square

Claim 3.5. *Suppose that Δ_λ holds for every $\omega \leq \lambda < \kappa$. Then Δ_κ holds.*

Proof. Fix $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ and \mathcal{E} , let $\text{cf}(\kappa) = \mu$ and fix a cofinal sequence of ordinals $(\kappa_\xi)_{\xi < \mu}$ in κ . Take a chain of elementary submodels $(M_\xi)_{\xi < \mu}$ such that everything relevant is in M_0 , $\kappa_\xi \subset M_\xi$ and $|M_\xi| = |\kappa_\xi|$ for $\xi < \mu$. The following is an easy consequence of M_ξ being elementary and X being Lindelöf:

Subclaim 3.5.1. *$(\mathcal{A}_i \cap M_\xi, \mathcal{B}_i \cap M_\xi)$ are weakly good pairs of size at most $|\kappa_\xi|$ for all $i < n$.*

By induction on $\xi < \mu$ construct an increasing sequence of weakly good pairs $\{(\mathcal{A}^\xi, \mathcal{B}^\xi) : \xi < \mu\}$ such that

- (i) $\bigcup_{i < n} (\mathcal{A}_i \cap M_\xi) \subset \mathcal{A}^\xi \subset \mathbb{B}$ and $\bigcup_{i < n} (\mathcal{B}_i \cap M_\xi) \subset \mathcal{B}^\xi \subset \mathbb{B}$,
- (ii) $\mathcal{A}^\xi, \mathcal{B}^\xi$ has size $\leq |\kappa_\xi|$,
- (iii) $\mathcal{A}^\xi, \mathcal{B}^\xi$ weakly fills $\mathcal{E} \cap M_\xi$,
- (iv) $\mathcal{A}^\xi \cap \mathcal{B}_i = \emptyset, \mathcal{A}^\xi \cap \mathcal{D}_j = \emptyset$ and $\mathcal{B}^\xi \cap \mathcal{A}_i = \emptyset, \mathcal{B}^\xi \cap \mathcal{C}_j = \emptyset$.

This can be done using $\Delta_{|\kappa_\xi|}$ at stage ξ . First note that $\mathcal{A}^{<\xi} = \bigcup\{\mathcal{A}^\zeta : \zeta < \xi\}$ and $\mathcal{B}^{<\xi} = \bigcup\{\mathcal{B}^\zeta : \zeta < \xi\}$ are of size at most $|\kappa_\xi|$ and $(\mathcal{A}^{<\xi}, \mathcal{B}^{<\xi})$

is a weakly good pair. Also, the family

$$\{(\mathcal{A}^{<\xi}, \mathcal{B}^{<\xi}), (\mathcal{A}_i \cap M_\xi, \mathcal{B}_i \cap M_\xi); (\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$$

is pairwise disjoint. Hence $\Delta_{|\kappa_\xi|}$ implies that there is a weakly good pair $(\mathcal{A}^\xi, \mathcal{B}^\xi)$ from \mathbb{B} of size at most $|\kappa_\xi|$ which fills $\mathcal{E} \cap M_\xi$ and is pairwise disjoint from $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ while

$$\mathcal{A}^{<\xi} \cup \bigcup_{i < n} (\mathcal{A}_i \cap M_\xi) \subset \mathcal{A}^\xi$$

and

$$\mathcal{B}^{<\xi} \cup \bigcup_{i < n} (\mathcal{B}_i \cap M_\xi) \subset \mathcal{B}^\xi.$$

Note that $\Delta_{|\kappa_\xi|}$ was used to find the common extension of $n+1$ weakly good pairs such that this extension is disjoint from $n+k$ given weakly good pairs. Now define $\mathcal{A} = \bigcup \{\mathcal{A}^\xi : \xi < \zeta\}$ and $\mathcal{B} = \bigcup \{\mathcal{B}^\xi : \xi < \zeta\}$; $(\mathcal{A}, \mathcal{B})$ is the desired extension. \square

This finishes the proof the lemma. \square

Corollary 3.6. *Every T_3 (locally) Lindelöf space is base resolvable. In particular, every T_3 locally countable or locally compact space is base resolvable.*

Proof. Fix a base \mathbb{B} for a T_3 Lindelöf space X and consider the set \mathbb{P} of all weakly good pairs $(\mathcal{A}, \mathcal{B})$ from \mathbb{B} partially ordered by extension. Note that we can apply Zorn's lemma to \mathbb{P} by Observation 3.2; pick a maximal weakly good pair $(\mathcal{A}, \mathcal{B}) \in \mathbb{P}$. Lemma 3.3 implies that a maximal weakly good pair must weakly fill every $\overline{U} \subset V$ pair, hence both \mathcal{A} and \mathcal{B} are bases of X .

Given a T_3 locally Lindelöf space X with a base \mathbb{B} consider its one-point Lindelöfization $X^* = X \cup \{x^*\}$ with the base $\mathbb{B}^* = \mathbb{B} \cup \{U \subseteq X^* : U \text{ is open in } X^*, x^* \in U\}$. X^* is T_3 Lindelöf hence base resolvable; thus \mathbb{B}^* can be partitioned to two bases which clearly gives a partition of \mathbb{B} . \square

4. COMBINATORICS OF RESOLVABILITY

In this section, we will prove a combinatorial lemma which will be our next tool in showing that further classes of space are base resolvable.

Definition 4.1. *Let $\mathcal{A}, \mathcal{B} \subseteq P(X)$. We say that \mathcal{A} **fills** \mathcal{B} iff*

$$U = \bigcup \{V \in \mathcal{A} : V \subsetneq U\}$$

*for every $U \in \mathcal{B}$. \mathcal{A}, \mathcal{B} is called a **good pair** iff \mathcal{A}, \mathcal{B} are disjoint, \mathcal{A} fills \mathcal{B} and \mathcal{B} fills \mathcal{A} . \mathcal{A} is **self-filling** if \mathcal{A} fills \mathcal{A} .*

Note that $\mathcal{A} \subseteq P(X)$ generates a topology on X iff \mathcal{A} fills $\{\cap \mathcal{B} : \mathcal{B} \in [\mathcal{A}]^{<\omega}\}$ and covers X .

Definition 4.2. A self-filling family \mathcal{A} is **resolvable** iff there is a partition $\mathcal{A}_0, \mathcal{A}_1$ of \mathcal{A} such that \mathcal{A}_i fills \mathcal{A} for $i < 2$.

Lemma 4.3. Suppose that $\mathbb{B} \subseteq P(X)$ fills itself. Then the following are equivalent:

- (1) for every $U \in \mathbb{B}$ there is a good pair $(\mathbb{B}_0^U, \mathbb{B}_1^U)$ from \mathbb{B} such that $U = \cup \mathbb{B}_0^U = \cup \mathbb{B}_1^U$,
- (2) \mathbb{B} is resolvable.

Proof. (2) implies (1) is trivial.

Let \mathcal{P} be the set of all good pairs $(\mathbb{B}_0, \mathbb{B}_1)$ formed by elements of \mathbb{B} ; \mathcal{P} is partially ordered by $(\mathbb{B}_0, \mathbb{B}_1) \leq (\mathbb{B}'_0, \mathbb{B}'_1)$ iff $\mathbb{B}_i \subseteq \mathbb{B}'_i$ for $i < 2$. It is clear that every chain in (\mathcal{P}, \leq) has an upper bound hence, by Zorn's lemma, we can pick a \leq -maximal element $(\mathbb{B}_0, \mathbb{B}_1) \in \mathcal{P}$.

We claim that \mathbb{B}_i fills \mathbb{B} for $i < 2$. Pick any $U \in \mathbb{B}$ and consider the good pair $\mathbb{B}_0^U, \mathbb{B}_1^U$ with $U = \cup \mathbb{B}_0^U = \cup \mathbb{B}_1^U$. Define

$$\mathbb{B}'_i = \mathbb{B}_i \cup (\mathbb{B}_i^U \setminus \mathbb{B}_{1-i})$$

for $i < 2$. It is easy to see that $(\mathbb{B}'_0, \mathbb{B}'_1)$ forms a good pair which fills $\{U\}$. Also, $(\mathbb{B}_0, \mathbb{B}_1) \leq (\mathbb{B}'_0, \mathbb{B}'_1)$ thus by the maximality of $(\mathbb{B}_0, \mathbb{B}_1)$ we have that $\mathbb{B}'_i = \mathbb{B}_i$. This finishes the proof. \square

The first corollary is a direct application and shows that resolvability is preserved by unions.

Corollary 4.4. Suppose that \mathbb{B}_α is a resolvable self-filling family for each $\alpha < \kappa$. Then $\cup\{\mathbb{B}_\alpha : \alpha < \kappa\}$ is a resolvable self-filling family as well.

Corollary 4.5. Suppose that a self-filling family \mathbb{B} has the property that for every $U \in \mathbb{B}$ there is $\mathcal{U} \in [\mathbb{B} \setminus \{U\}]^{\leq \omega}$ such that $U = \cup \mathcal{U}$. Then \mathbb{B} is resolvable.

Proof. We apply Lemma 4.3: fix a $U \in \mathbb{B}$ and we build the good pair $\mathbb{B}_0^U, \mathbb{B}_1^U \subseteq \mathbb{B}$ covering U by induction of length ω . First pick disjoint well founded, countable covers of U denoted by $\mathbb{B}_0^0, \mathbb{B}_1^0$. Then in each step $n \in \omega$ pick countable well founded subfamilies $\mathbb{B}_0^n, \mathbb{B}_1^n$ from $\mathbb{B} \setminus \cup\{\mathbb{B}_i^j : i < 2, j < n\}$ such that they are disjoint and they both fill in a previously chosen member of $\cup\{\mathbb{B}_i^j : i < 2, j < n\}$. By a straightforward bookkeeping argument we can guarantee that $\mathbb{B}_i^U = \cup\{\mathbb{B}_i^n : n \in \omega\}$ (both covering U) is a good pair. \square

Corollary 4.6. *Locally countable or hereditarily Lindelöf spaces are base resolvable without assuming any separation axioms.*

Our next corollary establishes that every reasonable space admits a resolvable base.

Corollary 4.7. *Suppose that \mathbb{B} is a base closed to finite unions in a T_1 topological space. Then \mathbb{B} is resolvable.*

Proof. We apply Lemma 4.3 again: fix $U \in \mathbb{B}$ and we construct a good pair covering U . Fix an arbitrary strictly decreasing sequence $\{U_n : n \in \omega\} \subseteq \mathbb{B}$ such that $U_0 \subseteq U$. Let

$$\mathbb{B}_i^U = \{V \in \mathbb{B} \cap \mathcal{P}(U) : \exists k \in \omega : U_{2k+i} \subseteq V \text{ but } U_{2k-1+i} \not\subseteq V\}$$

for $i < 2$. $\mathbb{B}_0^U \cap \mathbb{B}_1^U = \emptyset$ and it is easy to see that the assumption on the base guarantees that $(\mathbb{B}_0^U, \mathbb{B}_1^U)$ is a good pair. \square

Corollary 4.8. *The set of all open sets in a T_1 topological space is resolvable.*

Corollary 4.9. *Under Martin's Axiom every space X of local size $< 2^\omega$ is base resolvable without assuming any separation axioms.*

Proof. We apply Lemma 4.3: fix $U \in \mathbb{B}$ and we construct a good pair covering U . Note that we can suppose that $|U| = \kappa < 2^\omega$ without loss of generality. Select $\mathbb{B}_U \in [\mathbb{B}]^\kappa$ which fills itself and $\cup \mathbb{B}_U = U$. Now consider the ccc partial order $\mathbb{P} = Fn(\mathbb{B}_U, 2, \omega)$, i.e. the set of all finite partial functions from \mathbb{B}_U to 2. Now consider

$$D_{x,V,i} = \{f \in \mathbb{P} : \text{there is } W \in f^{-1}(i) : x \in W \subset V\}$$

for $i < 2, x \in U$ and $V \in \mathbb{B}_U$; note that each $D_{x,V,i}$ is dense in \mathbb{P} . Hence there is a filter $G \subseteq \mathbb{P}$ which intersects $D_{x,V,i}$ for $i < 2, x \in U$ and $V \in \mathbb{B}_U$. Let $\mathbb{B}_i = \{V \in \mathbb{B}_U : (\cup G)(V) = i\}$ for $i < 2$ and note that $(\mathbb{B}_0, \mathbb{B}_1)$ is the desired good pair. \square

5. THINNING SELF FILLING FAMILIES

Let \mathbb{B} be a self filling family; note that \mathbb{B} is *redundant* in the sense that $\mathbb{B} \setminus \mathcal{U}$ still fills \mathbb{B} for a finite or more generally, a well founded family \mathcal{U} .

Definition 5.1. *We say that $\mathcal{U} \subseteq \mathbb{B}$ is negligible iff $\mathbb{B} \setminus \mathcal{U}$ still fills \mathbb{B} .*

Our aim in this section is to show that every self filling family \mathbb{B} contains a negligible subfamily of size $|\mathbb{B}|$. Note that a base \mathbb{B} for a space X is resolvable iff it contains a negligible subfamily $\mathcal{U} \subseteq \mathbb{B}$ such that \mathcal{U} is a base of X as well. We will make use of the following definitions:

Definition 5.2. $\mathcal{U} \subseteq \mathcal{P}(X)$ is weakly increasing iff there is a well order \prec of \mathcal{U} such that $A \prec B$ implies that $B \setminus A \neq \emptyset$.

Definition 5.3. If \mathbb{B} fills itself then let

$$L(U, \mathbb{B}) = \min\{|\mathcal{V}| : \mathcal{V} \subseteq \mathbb{B} \setminus \{U\}, U = \cup \mathcal{V}\}$$

for $U \in \mathbb{B}$.

Observation 5.4. Suppose that \mathbb{B} fills itself and $\mathcal{U} \subseteq \mathbb{B}$.

- (1) There is weakly increasing $\mathcal{U}' \subseteq \mathcal{U}$ such that $\cup \mathcal{U} = \cup \mathcal{U}'$.
- (2) If \mathcal{U} is weakly increasing then \mathcal{U} is well founded with respect to inclusion; in particular, \mathcal{U} is negligible.
- (3) If $\mathbb{B} \setminus \mathcal{U}$ fills \mathcal{U} then \mathcal{U} is negligible.

Our first proposition establishes the main result for regular $|\mathbb{B}|$.

Proposition 5.5. Suppose that \mathbb{B} fills itself, and $\kappa = |\mathbb{B}|$ is regular. Then \mathbb{B} contains a negligible family of size κ .

Proof. We can suppose that $L(U, \mathbb{B}) < \kappa$ for every $U \in \mathbb{B}$; otherwise we can find a weakly increasing subfamily of size κ which is negligible by (2) of Observation 5.4. It suffices to define an increasing sequence of disjoint subsets $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \kappa$ such that \mathcal{V}_ξ fills \mathcal{U}_ξ and $\mathcal{U}_{\xi+1} \setminus \mathcal{U}_\xi \neq \emptyset$; clearly, $\mathcal{U} = \cup\{U_\xi : \xi < \kappa\}$ is a negligible set of size κ in \mathbb{B} by (3) of Observation 5.4. Suppose we have $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \zeta$ as above for some $\zeta < \kappa$; then $\mathbb{B} \setminus \cup\{\mathcal{U}_\xi, \mathcal{V}_\xi : \xi < \zeta\} \neq \emptyset$ by κ being regular hence we can select $U_\zeta \in \mathbb{B} \setminus \cup\{\mathcal{U}_\xi, \mathcal{V}_\xi : \xi < \zeta\}$ and define

$$\mathcal{U}_\zeta = \{U_\zeta\} \cup \bigcup\{\mathcal{U}_\xi : \xi < \zeta\}.$$

Find $\mathcal{W} \subseteq \mathbb{B} \setminus \{U_\zeta\}$ of size $< \kappa$ such $\cup \mathcal{W} = U_\zeta$; define

$$\mathcal{V}_\zeta = \bigcup\{\mathcal{V}_\xi : \xi < \zeta\} \cup (\mathcal{W} \setminus \mathcal{U}_\zeta).$$

It is easy to show that \mathcal{V}_ζ fills \mathcal{U}_ζ ; see the proof of Lemma 4.3. □

Theorem 5.6. Suppose that \mathbb{B} fills itself. Then \mathbb{B} contains a negligible family of size $|\mathbb{B}|$.

Proof. We can suppose that $\mu = \text{cf}(\kappa) < \kappa = |\mathbb{B}|$ and that every weakly increasing sequence in \mathbb{B} is of size less than κ . Fix a cofinal strictly increasing sequence of regular cardinals κ_ξ in κ such that $\mu < \kappa_0$ and define

$$\mathbb{B}_\xi = \{U \in \mathbb{B} : L(U, \mathbb{B}) \leq \kappa_\xi\}.$$

If there is a ξ such that every weakly increasing sequence is of size less than κ_ξ then $\mathbb{B} = \mathbb{B}_\xi$; define a set mapping $F : \mathbb{B} \rightarrow [\mathbb{B}]^{<\kappa_\xi^+}$ such that $U = \cup F(U)$ where $F(U) \subseteq \mathbb{B} \setminus \{U\}$. As $\kappa_\xi^+ < \kappa$ we can apply Hajnal's

Set Mapping theorem (see Theorem 19.2 in [1]): there is an F -free set \mathcal{U} of size κ in \mathbb{B} , i.e. $F(U) \cap \mathcal{U} = \emptyset$ for all $U \in \mathcal{U}$; observe that \mathcal{U} is negligible as $\cup\{F(U) : U \in \mathcal{U}\} \subseteq \mathbb{B} \setminus \mathcal{U}$ fills \mathcal{U} .

From now on we suppose that there are arbitrarily large weakly increasing sequences in \mathbb{B} . It suffices to define increasing sequences $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \mu$ such that

- (i) $\mathcal{U}_\xi, \mathcal{V}_\xi$ are disjoint and $\kappa_\xi \leq |\mathcal{U}_\xi|$,
- (ii) \mathcal{V}_ξ fills \mathcal{U}_ξ .

Indeed, the union $\cup\{\mathcal{U}_\xi : \xi < \mu\}$ is negligible in \mathbb{B} of size κ . Suppose we defined $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \zeta$; let

$$\lambda = (|\cup\{\mathcal{U}_\xi \cup \mathcal{V}_\xi : \xi < \zeta\}| \cdot \kappa_\zeta)^+.$$

Note that $\lambda < \kappa$ thus we can pick a weakly increasing $\mathcal{W} \in [\mathbb{B}]^\lambda$; without loss of generality, we can suppose that \mathcal{W} is disjoint from $\cup\{\mathcal{U}_\xi \cup \mathcal{V}_\xi : \xi < \zeta\}$. Note that

$$\mathcal{W} = \cup\{\mathbb{B}_\delta \cap \mathcal{W} : \delta < \mu\}$$

and that $\mu < \text{cf}(\lambda) = \lambda$, hence there is $\delta < \mu$ such that $\mathcal{W}' = \mathcal{W} \cap \mathbb{B}_\delta$ has size λ . Define $\mathcal{U}_\zeta = \mathcal{W}' \cup \cup\{\mathcal{U}_\xi : \xi < \zeta\}$.

Now, for every $U \in \mathcal{W}'$ select $F(U) \in [\mathbb{B} \setminus \{U\}]^{\kappa_\delta}$ such that $U = \cup F(U)$. Define

$$\mathcal{V}_\zeta = \cup\{\mathcal{V}_\xi : \xi < \zeta\} \cup \cup\{F(U) : U \in \mathcal{W}'\} \setminus \mathcal{U}_\zeta.$$

Note that $\kappa_\zeta \leq |\mathcal{U}_\zeta| = \lambda$ and $|\mathcal{V}_\zeta| \leq \lambda \cdot \kappa_\delta < \kappa$. It is only left to prove that \mathcal{V}_ζ fills \mathcal{U}_ζ ; in fact, it suffices to show that \mathcal{V}_ζ fills \mathcal{W}' . Suppose that \prec is the well ordering witnessing that \mathcal{W}' is weakly increasing and suppose that there is a $U \in \mathcal{W}'$ which is not filled by \mathcal{V}_ζ ; we can suppose that U is \prec -minimal. Fix an $x \in U$ witnessing that \mathcal{V}_ζ does not fill U . Pick $V \in F(U)$ such that $x \in V \subset U$; if $V \in \mathcal{W}'$ then $V \prec U$, thus V is filled by \mathcal{V}_ζ by the minimality of U . This contradicts the choice of x , hence $V \notin \mathcal{W}'$. Thus $V \in \mathcal{V}_\zeta \cup \cup\{\mathcal{U}_\xi : \xi < \zeta\}$ which is filled by \mathcal{V}_ζ by the inductive hypothesis; this again contradicts the choice of x , which finishes the proof. \square

6. IRRESOLVABLE SELF FILLING FAMILIES

The aim of this section is to construct an irresolvable self filling family and deduce the existence of a non base resolvable T_0 topological space.

Given a partial order (\mathbb{P}, \leq) and $p, q \in \mathbb{P}$ let

$$[p, q] = \{r \in \mathbb{P} : p \leq r \leq q\}.$$

The key to our construction is the following definition:

Definition 6.1. *We say that a poset \mathbb{P} without maximal elements satisfies*

$$\mathbb{P} \rightarrow (I_\omega)_2^1$$

iff for every partition $\mathbb{P} = D_0 \cup D_1$ there is $i < 2$ and strictly increasing $\{p_n : n \in \omega\} \subseteq D_i$ such that $[p_0, p_n] \subseteq D_i$ for every $n \in \omega$. The negation is denoted by $\mathbb{P} \nrightarrow (I_\omega)_2^1$.

The above definition is motivated by the following:

Observation 6.2. *For any irresolvable self filling family $\mathbb{B} \subseteq \mathcal{P}(X)$ the partial order $\mathbb{P} = (\mathbb{B}, \supseteq)$ satisfies $\mathbb{P} \rightarrow (I_\omega)_2^1$.*

Proof. Consider a partition of $\mathbb{P} = (\mathbb{B}, \supseteq)$ into sets D_0, D_1 ; as \mathbb{B} is irresolvable, there is $i < 2$, $x \in X$ and $U \in D_i$ such $V \in D_i$ for every $V \in \mathbb{B}$ with $x \in V \subseteq U$. Pick a strictly decreasing sequence $\{V_n : n \in \omega\} \subseteq \mathbb{B}$ such that $x \in V_n \subseteq U$ for every $n \in \omega$; clearly, $[V_0, V_n] \subseteq D_i$ for every $n \in \omega$. \square

Our next aim is to find a partial order \mathbb{P} first with $\mathbb{P} \rightarrow (I_\omega)_2^1$; note that trees or $Fn(\kappa, 2)$ cannot satisfy $\mathbb{P} \rightarrow (I_\omega)_2^1$. Moreover:

Proposition 6.3. *$\mathbb{P} \nrightarrow (I_\omega)_2^1$ for every countable poset \mathbb{P} without maximal elements.*

Proof. Define a rank function rk_p by induction on a well founded subset of $U_p = \{q \in \mathbb{P} : p \leq q\}$ (for each $p \in \mathbb{P}$) as follows:

$$\begin{aligned} rk_p(p) &= 0, \\ rk_p(t) &= \sup\{rk_p(s) + 1 : s \in U_p, s < t\} \\ &\text{if } rk_p(s) \text{ is defined for all } s \in U_p, s < t. \end{aligned} \tag{6.1}$$

We will refer to rk_p as the p -rank. Also, let $\{I_n : n \in \omega\}$ enumerate all intervals $I = [p', p]$ in \mathbb{P} which contain an infinite chain and let $\mathbb{P} = \{p_n : n \in \omega\}$ denote a 1-1 enumeration.

By induction on $n \in \omega$ construct disjoint $P_{0,n}, P_{1,n} \subseteq \mathbb{P}$ such that

- (i) $P_{i,n}$ is a finite union of antichains for $i < 2$,
- (ii) $p_n \in \cup_{i < 2} P_{i,n}$ and there is $q \in P_{i,n}$ such that $p_n \leq q$ for each $i < 2$,
- (iii) $I_n \cap P_{i,n} \neq \emptyset$ for $i < 2$,
- (iv) for every strictly increasing chain $C = \{c_k : k \in \omega\} \subseteq P$ containing only well founded intervals such that $p_n \in C$ we have

$$\bigcup_{k \in \omega} [c_0, c_k] \cap P_{i,n} \neq \emptyset$$

for each $i < 2$.

It is easy to see that such a construction yields a partition $P_i = \cup\{P_{i,n} : n \in \omega\}$ witnessing $\mathbb{P} \rightarrow (I_\omega)_2^1$.

Suppose we constructed $P_{i,n-1}$ satisfying the above conditions; note that finitely many elements can be added to both $P_{0,n-1}$ and $P_{1,n-1}$ without violating (i), thus (ii) and (iii) are easy to satisfy; note that $I_n \setminus \cup_{i < 2} P_{i,n-1}$ is infinite as I_n contains an infinite chain.

It suffices to show the following to finish our proof:

Claim 6.4. *Fix $p \in \mathbb{P}$ and $A \subseteq \mathbb{P}$ which is covered by finitely many antichains. Then there is an antichain $B \subseteq \mathbb{P} \setminus A$ such that for every increasing chain $C = \{c_k : k \in \omega\} \subseteq P$ containing only well founded intervals with $p \in C$ we have*

$$\bigcup_{k \in \omega} [c_0, c_k] \cap B \neq \emptyset.$$

Proof. Let $Q = \{q \in \mathbb{P} \setminus A : [p, q] \text{ is well founded}\}$ and define q^+ to be the element minimizing rk_p on $[p, q] \setminus A$ for $q \in Q$; let $B = \{q^+ : q \in Q\}$. First note that B is an antichain. Now fix a strictly increasing chain $C = \{c_k : k \in \omega\} \subseteq P$ containing only well founded intervals with $p \in C$; note that there is $q \in C \setminus A$ such that $p < q$; also, $q \in Q$ by $[p, q]$ being well founded. Thus $q^+ \in \bigcup_{k \in \omega} [c_0, c_k] \cap B$. \square

To finish the proof of the theorem, apply the claim twice: to $A = \cup P_{i,n-1}$ and define $P_{0,n} = P_{0,n-1} \cup B$ and next to $A = P_{0,n} \cup P_{1,n-1}$ similarly. \square

We will call a countable strictly increasing sequence of elements of \mathbb{P} a *branch*; we say that a branch $x = (x_n)_{n \in \omega}$ goes above an element $p \in \mathbb{P}$ iff $p \leq x_n$ for some $n \in \omega$.

Theorem 6.5. *There is a partial order \mathbb{P} of size ω_1 without maximal elements such that $\mathbb{P} \rightarrow (I_\omega)_2^1$. Furthermore,*

- (1) *every $p \in \mathbb{P}$ has finitely many predecessors,*
- (2) *if $p \not\leq q$ in \mathbb{P} then there is a branch x in \mathbb{P} which goes above q but not p .*

Proof. Let us fix a function $c : [\omega_1]^2 \rightarrow \omega$ such that $c(\cdot, \zeta) : \zeta \rightarrow \omega$ is 1-1 for every $\zeta \in \omega_1$. It is easy to see that such functions satisfy the following:

Fact 6.6. *If $c(\cdot, \zeta) : \zeta \rightarrow \omega$ is 1-1 for every $\zeta \in \omega_1$ for some $c : [\omega_1]^2 \rightarrow \omega$ then for every uncountable, disjoint family $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ and $N \in \omega$ there are $a < b^1$ in \mathcal{A} such that $c(\xi, \zeta) > N$ for every $\xi \in a, \zeta \in b$.*

¹ $a < b$ iff $\xi < \zeta$ for all $\xi \in a, \zeta \in b$

Also, fix an enumeration $\{(y_\alpha, w_\alpha) : \omega \leq \alpha < \omega_1\}$ of all pairs of elements of $\omega_1 \times \omega$ such that $y_\alpha, w_\alpha \in \alpha \times \omega$.

We define $\mathbb{P} = (\omega_1 \times \omega, \leq)$ as follows: by induction on $\alpha \in L_1$ (where L_1 stands for the limit ordinals in ω_1) we construct a poset $\mathbb{P}_\alpha = ((\alpha + \omega) \times \omega, \leq_\alpha)$ with properties:

- (i) \mathbb{P}_α has no maximal elements and every $p \in \mathbb{P}_\alpha$ has finitely many predecessors,
- (ii) $\leq_\alpha \upharpoonright \beta = \leq_\beta$ for all $\beta < \alpha$,
- (iii) $(\xi, n) <_\alpha (\zeta, m)$ implies that $\xi < \zeta$ and $\max(n, c(\xi, \zeta)) < m$,
- (iv) there is $t_\alpha \in \mathbb{P}_\alpha$ such that $t <_\alpha t_\alpha$ if and only if $t \leq_\alpha y_\alpha$ or $t \leq_\alpha w_\alpha$,
- (v) if $p \not\leq q$ in \mathbb{P}_α then there is a branch x in \mathbb{P}_α which goes above q but not p .

We only sketch the inductive step: suppose that $y_\alpha = (\xi, n)$ and $w_\alpha = (\zeta, m)$. Now find $k \in \omega$ larger than n, m and $c(\nu, \alpha)$ for every $\nu \in \omega_1$ such that there is $s \leq y_\alpha$ or $s \leq w_\alpha$ with $s = (\nu, l)$ for some $l \in \omega$; this can be done by (i). Now define $t_\alpha = (\alpha, k)$ and \leq_α so that $t <_\alpha t_\alpha$ implies that $t \leq_\alpha y_\alpha$ or $t \leq_\alpha w_\alpha$. Extend \leq_α further so that \mathbb{P}_α has no maximal elements and satisfies (v); this can be done by "placing" copies of $2^{<\omega}$ above elements of $\mathbb{P}_\alpha \setminus \cup\{\mathbb{P}_\beta : \beta < \alpha\}$.

Let us define $\mathbb{P} = \cup\{\mathbb{P}_\alpha : \alpha < \omega_1\}$ and $\leq = \cup\{\leq_\alpha : \alpha < \omega_1\}$; observe that (\mathbb{P}, \leq) is well defined and trivially satisfies (1) and (2). In what follows, π_{ω_1} and π_ω denotes the projections from $\omega_1 \times \omega$ to the first and second coordinates respectively.

Claim 6.7. $\mathbb{P} \rightarrow (I_\omega)_2^1$.

Proof. Suppose that $\mathbb{P} = D_0 \cup D_1$; we can assume that D_0 and D_1 are both cofinal. Now suppose that there is no increasing chain with each interval in one of the D_i and reach a contradiction as follows. We will say that an interval $[s, t]$ in \mathbb{P} is *i-maximal* for some $i < 2$ if $[s, t] \subseteq D_i$ but $[s, t'] \not\subseteq D_i$ for every $t < t'$. Observe that for every $s \in D_i$ there is $t \in D_i$ such that $[s, t]$ is i-maximal; otherwise we can construct an increasing chain starting from s with each interval in D_i . Now construct increasing 4-element sequences $R_\alpha = \{\tilde{x}_\alpha \leq \tilde{y}_\alpha \leq \tilde{z}_\alpha \leq \tilde{w}_\alpha\} \subseteq \mathbb{P}$ for $\alpha < \omega_1$ such that

- (a) $[\tilde{x}_\alpha, \tilde{y}_\alpha] \subseteq \mathbb{P}_0$ is a 0-maximal interval,
- (b) $[\tilde{z}_\alpha, \tilde{w}_\alpha] \subseteq \mathbb{P}_1$ is a 1-maximal interval,
- (c) $\pi_{\omega_1} R_\alpha < \pi_{\omega_1} R_\beta$ if $\alpha < \beta$.

By passing to a subsequence of $\{R_\alpha : \alpha < \omega_1\}$ we can suppose that $\pi_\omega R_\alpha$ is independent of α ; let $N = \max \pi_\omega R_\alpha$. Find $\alpha < \beta$, using Fact 6.6, such that

$$c \upharpoonright [\pi_{\omega_1} R_\alpha, \pi_{\omega_1} R_\beta] > N.$$

Observe that $\tilde{x}_\alpha \not\leq \tilde{w}_\beta$ by $\pi_\omega w_\beta = N < c(\pi_{\omega_1} \tilde{x}_\alpha, \pi_{\omega_1} \tilde{w}_\beta)$ and (iii). Now find $\gamma < \omega_1$ such that $(y_\gamma, w_\gamma) = (\tilde{y}_\alpha, \tilde{w}_\beta)$ and consider $t_\gamma \in \mathbb{P}_\gamma$. We claim that t_γ is a minimal extension of \tilde{y}_α and \tilde{w}_β in the following sense:

- (1) $[\tilde{x}_\alpha, t_\gamma] = [\tilde{x}_\alpha, \tilde{y}_\alpha] \cup \{t_\gamma\}$,
- (2) $[\tilde{z}_\beta, t_\gamma] = [\tilde{z}_\beta, \tilde{w}_\beta] \cup \{t_\gamma\}$.

Indeed, if $\tilde{x}_\alpha \leq t' < t_\gamma$ then $t' \leq \tilde{y}_\alpha$ or $t' \leq \tilde{w}_\beta$; $\tilde{x}_\alpha \not\leq \tilde{w}_\beta$ implies that $t' \not\leq w_\beta$ hence $t' \in [\tilde{x}_\alpha, \tilde{y}_\alpha]$. Similarly, if $\tilde{z}_\beta \leq t' < t_\gamma$ then $t' \leq \tilde{y}_\alpha$ or $t' \leq \tilde{w}_\beta$; however, $t' \not\leq \tilde{y}_\alpha$ by $\pi_\omega t' > \pi_\omega \tilde{y}_\alpha$ so $t' \in [\tilde{z}_\beta, \tilde{w}_\beta]$.

Note that $t \in \mathbb{P}_0$ contradicts the 0-maximality of $[\tilde{x}_\alpha, \tilde{y}_\alpha]$ and (1) while $t \in \mathbb{P}_1$ contradicts the 1-maximality of $[\tilde{z}_\beta, \tilde{w}_\beta]$ and (2). \square

The above claim finishes the proof. \square

Using the previous theorem, we construct an irresolvable self-filling family; we can actually realize this family as a system of open sets in a first countable compact space. We remark that this space is base resolvable, as every compact space, by Corollary 3.6.

Theorem 6.8. *There is a first countable Corson compact space (X, τ) and $\mathcal{U} \subseteq \tau$ such that \mathcal{U} fills $\{\cap \mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{<\omega}\}$ and \mathcal{U} is irresolvable.*

Proof. Consider the poset \mathbb{P} in Theorem 6.5. We say that $x \in \mathbb{P}^\omega$ is a *maximal chain* iff $(x(n))_{n \in \omega}$ is a branch in \mathbb{P} , $x(0)$ is a minimal element of \mathbb{P} and $[x(n), x(n+1)] = \{x(n), x(n+1)\}$. Note that there are no increasing chains of order type $\omega + 1$ in \mathbb{P} . Furthermore

Observation 6.9. (1) *Any branch $y \in \mathbb{P}^\omega$ can be extended to a maximal chain $\bar{y} \in \mathbb{P}^\omega$,*
 (2) *there is an $n_0 \in \omega$ such that $\cup_{n_0 \leq n} [\bar{y}(n_0), \bar{y}(n)] \subseteq \cup_{n \in \omega} [y(0), y(n)]$.*

Note that (2) implies that if $y \in \mathbb{P}^\omega$ has homogeneous intervals with respect to some coloring of \mathbb{P} then the an end-segment of the maximal extension \bar{y} has the same property.

Now consider $X = \{x \in \mathbb{P}^\omega : x \text{ is a maximal chain}\}$ as a subspace of $2^\mathbb{P}$; here $2^\mathbb{P}$ is equipped with the usual product topology.

Claim 6.10. *X is a compact subspace of $\Sigma(2^\mathbb{P}) = \Sigma(2^{\omega_1})$.*

Proof. $\Sigma(2^\mathbb{P}) = \Sigma(2^{\omega_1})$ follows from $|\mathbb{P}| = \omega_1$ and clearly every chain is countable so $X \subseteq \Sigma(2^\mathbb{P})$.

We prove that X is a closed subset of $2^\mathbb{P}$. Suppose that $y \in 2^\mathbb{P} \setminus X$; clearly, if y is not a chain then y can be separated from X . Suppose that y is a chain, then either $y(0)$ is not minimal in \mathbb{P} or there is $n \in \omega$ such that $(y(n), y(n+1)) \neq \emptyset$. In the first case let $\varepsilon \in Fn(\mathbb{P}, 2)$ be defined to be 1 on $y(0)$ and $\varepsilon(p) = 0$ for $p < y(0)$, $p \in \mathbb{P}$ (note that each element in \mathbb{P} has only finitely many predecessors); then $y \in [\varepsilon]$ and $[\varepsilon] \cap X = \emptyset$.

In the second case let $\varepsilon \in Fn(\mathbb{P}, 2)$ such that $1 = \varepsilon(y(n)) = \varepsilon(y(n+1))$ and $\varepsilon \upharpoonright (y(n), y(n+1)) = 0$; then $y \in [\varepsilon]$ and $[\varepsilon] \cap X = \emptyset$. \square

Claim 6.11. $\{x\} = \cap\{[\chi_{x(n)}] \cap X : n \in \omega\}$ for every $x \in X$. Hence every point in X has countable Ψ -character; in particular, X is first countable.

Proof. Suppose that $y \in \cap\{[\chi_{x(n)}] \cap X : n \in \omega\}$, that is $\{x(n) : n \in \omega\} \subset \{y(n) : n \in \omega\}$. We prove that $x(n) = y(n)$ by induction on $n \in \omega$. $y(0) = x(0)$ as they are both minimal elements in \mathbb{P} . Suppose that $x(i) = y(i)$ for $i < n$; if $x(n) \neq y(n)$ then $x(n) = y(k)$ for some $n < k$, thus $y(n) \in (x(n-1), x(n)) = (y(n-1), y(k))$ which contradicts the maximality of the chain x . \square

Now define

$$V_p = \{x \in X : \exists n \in \omega : x(n) \geq p\} \text{ for } p \in \mathbb{P},$$

and note that V_p is open since $V_p = \cup\{[\chi_{\{q\}}] \cap X : p \leq q\}$. We define $\mathcal{U} = \{V_p : p \in \mathbb{P}\}$.

Claim 6.12. \mathcal{U} is an irresolvable self filling family.

Proof. Note that $p < q$ in \mathbb{P} if and only if $V_q \subsetneq V_p$; the nontrivial direction is implied by property (2) of \mathbb{P} in Theorem 6.5. Now it is easy to see that \mathcal{U} fills itself.

We show that \mathcal{U} is irresolvable; suppose that we partitioned \mathcal{U} , equivalently \mathbb{P} into two parts $\mathbb{P}_0, \mathbb{P}_1$. Applying $\mathbb{P} \rightarrow (I_\omega)_2^1$ we see that there is a chain $y \in \mathbb{P}^\omega$ and $i < 2$ such that $[y(0), y(n)] \subseteq \mathbb{P}_i$ for every $n \in \omega$. By our previous Observation 6.9 there is $\bar{y} \in X$ such that $[\bar{y}(n_0), \bar{y}(n)] \subseteq \mathbb{P}_i$ for some $n_0 \in \omega$ and every $n \geq n_0$. We claim that there is no $V \in \{V_p : p \in \mathbb{P}_{1-i}\}$ such that $\bar{y} \in V \subseteq V_{\bar{y}(n_0)}$. Indeed, if $\bar{y} \in V_p \subseteq V_{\bar{y}(n_0)}$ for some $p \in \mathbb{P}$ then $\bar{y}(n_0) \leq p$ and there is $n \in \omega \setminus n_0$ such that $p \leq \bar{y}(n)$; that is $p \in [\bar{y}(n_0), \bar{y}(n)] \subseteq \mathbb{P}_i$. \square

The last claim finishes the proof of the theorem. \square

Let us finish this section with the following:

Corollary 6.13. *There is a non base resolvable, T_0 topological space.*

Proof. There is an irresolvable self filling family $\mathcal{U} \subseteq \mathcal{P}(X)$ (on some set X) such that \mathcal{U} fills $\{\cap \mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{<\omega}\}$ by Theorem 6.8. Define a relation \sim on X by $x \sim y$ iff $\{U \in \mathcal{U} : x \in U\} = \{U \in \mathcal{U} : y \in U\}$; clearly, \sim is an equivalence relation on X . Let $[x]$ stand for the \sim -class of $x \in X$; let $[U] = \{[x] : x \in U\}$ and note that $[\mathbb{B}] = \{[U] : U \in \mathcal{U}\}$ is a base for a T_0 topology on $[X]$. It is easy to see that $[\mathbb{B}]$ is an irresolvable base. \square

7. A 0-DIMENSIONAL, HAUSDORFF SPACE WITH AN IRRESOLVABLE BASE

In this section, we significantly strengthen Corollary 6.13 by showing

Theorem 7.1. *It is consistent that there is a first countable, 0-dimensional, T_2 space which has a point countable, irresolvable base. Furthermore, the space has size \mathfrak{c} and weight ω_1 .*

Proof. For $\langle \alpha, n \rangle, \langle \beta, m \rangle \in \omega_1 \times \omega$ write $\langle \alpha, n \rangle \triangleleft \langle \beta, m \rangle \in \omega_1 \times \omega$ iff $\langle \alpha, n \rangle = \langle \beta, m \rangle$ or $(\alpha < \beta \text{ and } n < m)$.

Definition 7.2. *If $\preceq_1, \preceq_2 \subset \triangleleft$, then let $\preceq_1 \sqcup \preceq_2$ be the partial order generated by $\preceq_1 \cup \preceq_2$.*

Definition 7.3. *If $\mathcal{A} = \langle \omega_1 \times \omega, \preceq \rangle$ is a poset with $\preceq \subset \triangleleft$, and for each $\alpha \in L_1$ we have a set $T_\alpha \subset \alpha \times \omega$ such that*

(C) $\langle T_\alpha, \preceq \rangle$ is an everywhere ω -branching tree,

then we say that the pair $\langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1 \rangle \rangle$ is a candidate.

Denote $T_\alpha(n)$ the n^{th} level of the tree $\langle T_\alpha, \preceq \rangle$.

Definition 7.4. *Fix a candidate $\mathbb{A} = \langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1 \rangle \rangle$. We will define a topological space $X(\mathbb{A})$ as follows.*

For $\alpha \in L_1$ let $B(T_\alpha)$ be the collection of the cofinal branches of T_α , and let

$$\mathcal{B}(\mathbb{A}) = \bigcup \{B(T_\alpha) : \alpha \in L_1\}.$$

The underlying set of the space $X(\mathbb{A})$ is $\mathcal{B}(\mathbb{A})$.

For $x \in \omega_1 \times \omega$ let $U(x) = \{y \in \omega_1 \times \omega : x \preceq y\}$ and

$$V(x) = \{b \in \mathcal{B}(\mathbb{A}) : \exists y \in b (x \preceq y)\}.$$

Clearly $V(x) = \{b \in \mathcal{B}(\mathbb{A}) : b \subseteq^ U(x)\}$.*

We declare that the family

$$\mathcal{V} = \{V(x) : x \in \omega_1 \times \omega\}$$

is the base of $X(\mathbb{A})$.

Lemma 7.5. *\mathcal{V} is a base, and so $X(\mathbb{A})$ is a topological space.*

Proof. Assume that $b \in V(x) \cap V(y)$. Then there is $z \in b$ such that $x \preceq z$ and $y \preceq z$. Then $b \in V(z) \subset V(x) \cap V(y)$. \square

For $x, y \in \omega_1 \times \omega$ with $x \preceq y$ let

$$[x, y] = \{t \in \omega_1 \times \omega : x \preceq t \preceq y\}.$$

Definition 7.6. *We say that a candidate $\mathbb{A} = \langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1 \rangle \rangle$ is good iff*

- (G1) $V(u) \supset V(v)$ iff $u \preceq v$.
 (G2) $\forall \alpha \in L_1 \forall \zeta < \alpha (T_\alpha \setminus (\zeta \times \omega)) \neq \emptyset$.
 (G3) (a) $\forall \alpha \in L_1 (\forall x, y \in T_\alpha) U(x) \cap U(y) \neq \emptyset$ iff x and y are \preceq -comparable.
 (b) for each $\{\alpha, \beta\} \in [L_1]^2$ there is $f(\alpha, \beta) \in \omega$ such that
 $\forall x \in T_\alpha(f(\alpha, \beta)) \forall y \in T_\beta(f(\alpha, \beta)) U(x) \cap U(y) = \emptyset$.
 (G4) For each $x \in \omega_1 \times \omega$ and $\alpha \in L_1$ there is $g(x, \alpha) \in \omega$ such that for each $y \in T_\alpha(g(x, \alpha))$

$$U(y) \subset U(x) \text{ or } U(y) \cap U(x) = \emptyset.$$

- (G5) If for all $\alpha \in L_1$ and $\zeta < \alpha$ we choose a four element \prec -increasing sequence

$$\langle x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha \rangle \subset T_\alpha \setminus (\zeta \times \omega)$$

then there are $\{\alpha, \beta\} \in [L_1]^2$, $\zeta < \alpha$, $\xi < \beta$, and $t \in T_\alpha \cap T_\beta$ such that

- (i) $y_\zeta^\alpha \prec t$ and $[x_\zeta^\alpha, t] = [x_\zeta^\alpha, y_\zeta^\alpha] \cup \{t\}$,
 (ii) $w_\xi^\beta \prec t$ and $[z_\xi^\beta, t] = [z_\xi^\beta, w_\xi^\beta] \cup \{t\}$.

Lemma 7.7. *If \mathbb{A} is a good candidate, then $X(\mathbb{A})$ is a dense-in-itself, first countable, 0-dimensional T_2 space such that the base $\{V(x) : x \in \omega_1 \times \omega\}$ is point countable and irresolvable.*

Proof. We prove this lemma in several steps.

Claim 7.8. $X(\mathbb{A})$ is dense-in-itself.

Indeed, assume that $b \in B(T_\alpha)$ and $V(x)$ is an open neighbourhood of b . Then there is $y \in b$ with $x \preceq y$ and so $b \in V(y) \subset V(x)$. Thus $V(x) \supset V(y) \supset \{b' \in B(T_\alpha) : y \in b'\}$, and so $V(x)$ has 2^ω many elements. So b is not isolated.

Claim 7.9. $X(\mathbb{A})$ is T_2 .

Indeed, let $b \in B(T_\alpha)$ and $c \in B(T_\beta)$.

If $\alpha = \beta$ then pick n such that x , the n^{th} element of b , and y , the n^{th} element of c , are different. Then $b \in V(x)$, $c \in V(y)$ and $V(x) \cap V(y) = \emptyset$ by (G3)(a).

If $\alpha \neq \beta$ then write $n = f(\alpha, \beta)$ (see G3)(b)), let x be the n^{th} element of b , and let y be the n^{th} element of c . Then $b \in V(x)$, $c \in V(y)$ and $V(x) \cap V(y) = \emptyset$ by (G3)(b).

Claim 7.10. $X(\mathbb{A})$ is 0-dimensional.

Indeed, assume that $x \in \omega_1 \times \omega$, $b \in \mathcal{B}(T_\alpha)$ and $b \notin V(x)$. Let $\{y\} = b \cap T_\alpha(g(\alpha, x))$. Then $y \notin U(x)$ because $b \notin V(x)$, so $U(x) \cap U(y) = \emptyset$ by (G4). Thus $V(x) \cap V(y) = \emptyset$ as well.

Claim 7.11. *The base $\{V(x) : x \in \omega_1 \times \omega\}$ is irresolvable.*

Assume on the contrary that there is a partition (K_0, K_1) of $\omega_1 \times \omega$ such that both $\mathcal{V}_0 = \{V(x) : x \in K_0\}$ and $\mathcal{V}_1 = \{V(x) : x \in K_1\}$ are bases.

Assume that $\alpha \in L_1$, $x, y \in T_\alpha$ with $x \preceq y$ and $i \in 2$. We say that interval $[x, y]$ is *i-maximal* in T_α iff

- (i) $[x, y] \subset K_i$, but $[x, z] \not\subset K_i$ for any $y \prec z \in T_\alpha$.

Subclaim 7.11.1. *If $\alpha \in L_1$ and $x \in T_\alpha \cap K_i$, then there is $x \preceq y \in T_\alpha$ such that the interval $[x, y]$ is K_i -maximal in T_α .*

Proof of the Claim. Assume on the contrary that there is no such y . Then we can construct a strictly increasing sequence $\langle x, y_0, y_1, \dots \rangle$ in T_α such that $[x, y_n] \subset K_i$ for all $n < \omega$.

Then $b = \{y \in T_\alpha : \exists n \in \omega \ y \preceq y_n\} \in \mathcal{B}(T_\alpha)$.

Since $b \in V(x)$, and we assumed that $\{V(z) : z \in K_{1-i}\}$ is a base, there is $z \in K_{1-i}$ with $b \in V(z) \subset V(x)$. Then $x \preceq z$ by (G1). Moreover, there is $y \in b$ with $z \prec y$ because $b \in V(z)$. Thus $z \in [x, y] \cap K_{1-i}$, so $[x, y] \not\subset K_i$. Contradiction, the subclaim is proved. \square

Using the subclaim, for all $\alpha \in L_1$ and for all $\zeta < \alpha$ we will construct a four element \preceq -increasing sequence

$$\langle x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha \rangle \subset T_\alpha \setminus (\zeta \times \omega)$$

as follows.

First, using (G2) pick $s_\zeta^\alpha \in T_\alpha \setminus (\zeta \times \omega)$.

If $K_0 \cap U(s_\zeta^\alpha) \cap T_\alpha = \emptyset$, then let $x_\zeta^\alpha = y_\zeta^\alpha = s_\zeta^\alpha$.

Otherwise pick

$$x_\zeta^\alpha \in K_0 \cap U(s_\zeta^\alpha) \cap T_\alpha,$$

and then, using the Subclaim above, pick

$$y_\zeta^\alpha \in U(x_\zeta^\alpha) \cap T_\alpha$$

such that

$$[x_\zeta^\alpha, y_\zeta^\alpha] \text{ is 0-maximal in } T_\alpha.$$

If $K_1 \cap U(y_\zeta^\alpha) \cap T_\alpha = \emptyset$, then let $z_\zeta^\alpha = w_\zeta^\alpha = y_\zeta^\alpha$.

Otherwise pick

$$z_\zeta^\alpha \in K_1 \cap U(y_\zeta^\alpha) \cap T_\alpha,$$

and then, using the Subclaim above, pick

$$w_\zeta^\alpha \in U(z_\zeta^\alpha) \cap T_\alpha$$

such that

$$[z_\zeta^\alpha, w_\zeta^\alpha] \text{ is 1-maximal in } T_\alpha.$$

By (G5), there are $\{\alpha, \beta\} \in [L_1]^2$, $\zeta < \alpha$, $\xi < \beta$, and $t \in T_\alpha \cap T_\beta$ such that

- (i) $y_\zeta^\alpha \prec t$ and $[x_\zeta^\alpha, t] = [x_\zeta^\alpha, y_\zeta^\alpha] \cup \{t\}$,
- (ii) $w_\xi^\beta \prec t$ and $[z_\xi^\beta, t] = [z_\xi^\beta, w_\xi^\beta] \cup \{t\}$.

Assume first that $t \in K_0$. Then $t \in K_0 \cap T_\alpha$, and $[x_\zeta^\alpha, t] = [x_\zeta^\alpha, y_\zeta^\alpha] \cup \{t\}$, so $[x_\zeta^\alpha, t] \subset K_0$, i.e. $[x_\zeta^\alpha, y_\zeta^\alpha]$ was not 0-maximal in T_α . Contradiction. If $t \in K_1$, then a similar argument works using the interval $[z_\xi^\beta, w_\xi^\beta]$ and K_1 .

So in both cases we obtained a contradiction, so the base $\{V(x) : x \in \omega_1 \times \omega\}$ is irresolvable, which proves the lemma. \square

Next we show that some c.c.c. forcing introduces a good candidate.

Define the poset $\mathcal{P} = \langle P, \leq \rangle$ as follows. The underlying set consists of 6-tuples

$$\langle A, \preceq, I, \{T_\alpha : \alpha \in I\}, f, g \rangle,$$

where

- (P1) $A \in [\omega_1 \times \omega]^{<\omega}$, $\langle A, \preceq \rangle$ is a poset, $\preceq \subset \triangleleft$, $I \in [\omega_1]^{<\omega}$,
- (P2) $T_\alpha \subset (A \cap \alpha) \times \omega$ and $\langle T_\alpha, \preceq \rangle$ is a tree for $\alpha \in I$,
- (P3) f and g are functions, $\text{dom}(f) \subset [I]^2$, $\text{dom}(g) \subset A \times I$, $\text{ran}(f) \cup \text{ran}(g) \subset \omega$
- (P4) To simplify our notation write $U(x) = \{y \in A : x \preceq y\}$ for $x \in A$.
 - (a) If $\alpha \in I$ and $x, y \in T_\alpha$ then $U(x) \cap U(y) \neq \emptyset$ iff x and y are \preceq -comparable.
 - (b) If $\{\alpha, \beta\} \in [\text{dom}(f)]^2$ and $n = f(\alpha, \beta)$, then

$$U[T_\alpha(n)] \cap U[T_\beta(n)] = \emptyset \text{ and } U[T_\alpha(n)] \cap T_\beta(< n) = \emptyset.$$

- (P5) if $\langle x, \alpha \rangle \in \text{dom}(g)$ then for all $y \in T_\alpha(g(x, \alpha))$ we have $U(y) \subset U(x)$ or $U(y) \cap U(x) = \emptyset$.

For $p \in P$ write $p = \langle A^p, \preceq^p, I^p, \{T_\alpha^p : \alpha \in I^p\}, f^p, g^p \rangle$, and for $x \in A^p$ let $U^p(x) = \{y \in A^p : x \preceq^p y\}$.

For $p, q \in P$ let $p \leq q$ iff

- (O1) $A^p \supset A^q$, and $\preceq^q = \preceq^p \upharpoonright A^q$,
- (O2) $I^p \supset I^q$ and $T_\alpha^q = T_\alpha^p \cap A^q$ for $\alpha \in I^q$,
- (O3) if $x \in A^p \setminus A^q$, then $U^p(x) \cap A^q = \emptyset$,
- (O4) $f^p \supset f^q$ and $g^p \supset g^q$,
- (O5) if $U^q(x) \cap U^q(y) = \emptyset$ then $U^p(x) \cap U^p(y) = \emptyset$.

Clearly \leq is a partial order on P .

For $p \in P$ write $\text{supp}(p) = I^p \cup \{\alpha : \langle \alpha, n \rangle \in A^p \text{ for some } n \in \omega\}$.

If \mathcal{G} is a \mathcal{P} -generic filter, then let

$$\begin{aligned} A &= \bigcup \{A^p : p \in \mathcal{G}\}, \\ \preceq &= \bigcup \{\preceq^p : p \in \mathcal{G}\}, \\ I &= \bigcup \{I^p : p \in \mathcal{G}\}, \\ T_\alpha &= \bigcup \{T_\alpha^p : \alpha \in p \in \mathcal{G}\} \text{ for } \alpha \in L_1, \\ f &= \bigcup \{f^p : p \in \mathcal{G}\}, \\ g &= \bigcup \{g^p : p \in \mathcal{G}\}. \end{aligned}$$

We show that \mathcal{P} satisfies c.c.c, and $\mathbb{A} = \langle \langle \omega_1 \times \omega, \preceq \rangle, \{T_\alpha : \alpha \in L_1\} \rangle$ is a good candidate.

Definition 7.12. *We say that the conditions p and q are twins iff*

(T1) $|\text{supp}(p)| = |\text{supp}(q)|$, moreover $\max(\text{supp}(p) \cap \text{supp}(q)) < \min(\text{supp}(p) \Delta \text{supp}(q))$,

Denote ρ the unique order preserving bijection between $\text{supp}(p)$ and $\text{supp}(q)$, and define the function $\underline{\rho} : \text{supp}(p) \times \omega \rightarrow \text{supp}(q) \times \omega$ by the formula $\underline{\rho}(\langle \alpha, n \rangle) = (\langle \rho(\alpha), n \rangle)$.

- (T2) $\underline{\rho}'' A^p = A^q$
- (T3) $x \preceq^p y$ iff $\underline{\rho}(x) \preceq^q \underline{\rho}(y)$
- (T4) $\rho'' I^p = I^q$
- (T5) $T_{\rho(\alpha)}^q = \underline{\rho}'' T_\alpha^p$.
- (T6) $f^p(x, y) = m$ iff $f^q(\underline{\rho}(x), \underline{\rho}(y)) = m$,
- (T7) $g^p(x, \alpha) = m$ iff $g^q(\underline{\rho}(x), \rho(\alpha)) = m$.

Lemma 7.13. *If p and q are twins then*

$$\begin{aligned} p \oplus q &= \\ \langle A^p \cup A^q, \preceq^p \cup \preceq^q, I^p \cup I^q, \{T_\alpha^p \cup T_\alpha^q : \alpha \in I^p \cup I^q\}, f^p \cup f^q, g^p \cup g^q \rangle \end{aligned}$$

is a common extension of p and q , where $T^p(\alpha) = \emptyset$ for $\alpha \notin I^p$.

Proof. Straightforward. □

Lemma 7.14. *There is a function φ from P into some countable set such that if $\varphi(p) = \varphi(q)$ and $\text{supp}(p) \cap \text{supp}(q) < \text{supp}(p) \Delta \text{supp}(q)$, then p and q are twins.*

Proof. Let $\varphi(p)$ be the type of the first order structure

$$\langle \text{supp}(p) \times \omega, A^p, \preceq^p, I^p, \{T_\alpha^p : \alpha \in I^p\}, f^p, g^p \rangle.$$

□

Lemmas 7.13 and 7.14 yield that \mathcal{P} satisfies c.c.c

Lemma 7.15. *$A = \omega_1 \times \omega$, $I = L_1$ and $T_\gamma(0) \setminus (\zeta \times \omega)$ is infinite for all $\gamma \in L_1$ and $\zeta < \gamma$, and so (G2) holds.*

Proof. For $p \in P$, $\gamma \in L_1$ and $y \in (\gamma \times \omega) \setminus A^p$ define $p \uplus \{y\}_\gamma$ as follows:

$$p \uplus \{y\}_\gamma = \langle A^p \cup \{y\}, \preceq^p, I^p \cup \{\gamma\}, \{T_\gamma^p \cup \{y\}, T_\alpha^p : \alpha \in I^p \setminus \{\gamma\}\}, f^p, g^p \rangle.$$

Then $q = p \uplus \{y\}_\gamma \in P$ and $p \uplus \{y\}_\gamma \leq p$. If we pick $y \notin \zeta \times \omega$, then $q \Vdash y \in T_\gamma \setminus (\zeta \times \omega)$, so we are done. □

Lemma 7.16. (a) *Assume that $p \in P$, $a \in T_\gamma^p$, and $b \in (\gamma \times \omega) \setminus A^p$ with $a \triangleleft b$. Let*

$$p \uplus_a \{b\}_\gamma = \langle A^p \cup \{b\}, \preceq^p \cup \{\langle a, b \rangle\}, \{T_\gamma^p \cup \{b\}, T_\alpha^p : \alpha \in I^p \setminus \{\gamma\}\}, f^p, g^p \rangle.$$

Then $p \uplus_a \{b\}_\gamma \in P$ and $p \uplus_a \{b\}_\gamma \leq p$.

(b) *The structure \mathbb{A} is a candidate.*

Proof. First we check $q = p \uplus_a \{b\}_\gamma \in P$.

(P1)-(P3) are straightforward.

(P4)(a): Since $U^q(b) = \{b\}$, we can assume that $x, y \neq b$. If $U^p(x) \cap U^p(y) \neq \emptyset$ then x and y are \preceq^p -comparable. So we can assume that $b \in U^q(x) \cap U^q(y)$. But then $a \in U^p(x) \cap U^p(y)$, so we are done.

(P4)(b): Assume that $x \in T_\alpha^q(n)$, $y \in T_\beta^q(n)$ and $z \in U^q(x) \cap U^q(y)$. If $z \neq b$, then $z \in U^p(x) \cap U^p(y)$ which is not possible. So $z = b$.

If $x, y \neq b$, then $a \in U^p(x) \cap U^p(y)$ which is not possible. So we can assume that $x = b$ and $\alpha = \gamma$. So $b \in T_\alpha^q(n)$ and so $a \in T_\alpha^p(n-1)$. Thus $T_\alpha^p(n-1) \cap U^p(y) \neq \emptyset$ which is not possible because (P4)(b) holds for p .

Assume that $x \in T_\alpha^q(n)$, $y \in T_\beta^q(< n)$ and $y \in U^q(x)$. If $y \neq b$, then $y \in U^p(x) \cap T_\beta^p(< n)$ which is not possible. So $y = b$ and $\beta = \gamma$. Thus $a \in T_\beta^p(< n) \cap U_\alpha^p(x)$ which is not possible because (P4)(b) holds for p .

(P5) Since $U(b) = \{b\}$, we can assume that $y \in A^p$. Since $b \in U^q(z)$ iff $a \in U^q(z)$ for $z \in A^p$, if $U^p(y) \subset U^p(x)$ then $U^q(y) \subset U^q(x)$, and if $U^p(y) \cap U^p(x) = \emptyset$ then $U^q(y) \cap U^q(x) = \emptyset$.

Thus we proved $q \in P$. Since $q \leq p$ is straightforward, we are done.
 (b) is clear from (a) by standard density arguments. \square

Lemma 7.17. \mathbb{A} has property (G1).

Proof. Assume that $p \in P$, $u, v \in A^p$, $v \notin U^p(u)$. Pick $\gamma \in L_1 \setminus I^p$ with $\text{supp}(p) \subset \gamma$, and pick $b \in \gamma \times \omega$ with $v \triangleleft b$.

Consider the condition $q = p \uplus_v \{b\}_\gamma \leq p$.

Since $b \in T_\gamma^q$, we have $V(b) \cap \mathcal{B}(T_\gamma) \neq \emptyset$, so $V(b) \neq \emptyset$. Since $U^q(u) \cap U^q(b) = \emptyset$ we have $U(u) \cap U(b) = \emptyset$, and so $V(u) \cap V(b) = \emptyset$, and so $\emptyset \neq V(b) \subset V(v) \setminus V(u)$. \square

Lemma 7.18. $\text{dom}(f) = [L_1]^2$ and $\text{dom}(g) = \omega_1 \times \omega \times L_1$. Hence (G3) and (G4) hold.

Proof. Assume that $\{\gamma, \delta\} \in [I^p]^2 \setminus \text{dom}(f^p)$.

Pick m such that $T_\alpha^p(m) = \emptyset$ for all $\alpha \in I^p$.

Extends f^p to f^q as follows: $\text{dom}(f^q) = \text{dom}(f^p) \cup \{\{\gamma, \delta\}\}$ and $f^q(\gamma, \delta) = m$.

Let

$$q = \langle A^p, \preceq^p, I^p, \{A_\alpha^p : \alpha \in I^p, f^q, g^p\} \rangle.$$

Then $q \in P$ and $q \leq p$.

Similar argument works for g . \square

Finally we verify that (G5) also holds.

Assume that

$$V^P \models \forall \alpha \in L_1 \forall \zeta < \alpha$$

$$\langle x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha \rangle \subset T_\alpha \setminus (\zeta \times \omega) \text{ is } \preceq\text{-increasing.}$$

For all $\alpha \in L_1$ and $\zeta < \alpha$ pick a condition $p_\zeta^\alpha = \langle A_\zeta^\alpha, \preceq_\zeta^\alpha, \dots \rangle$ which decides the sequence $\langle x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha \rangle$ and $\{x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha\} \subset T_\zeta^\alpha$.

Let us say that a Δ -system $\mathcal{A} \subset [\omega]^{<\omega}$ is *nice* iff $A \cap B < A \triangle B$ for all $A \neq B \in \mathcal{A}$.

Using the Fodor lemma, for each $\zeta \in \omega_1$ find $m_\zeta < \omega$ and $I_\zeta \in [L_1]^{\omega_1}$ such that

- (i) $\varphi(p_\zeta^\alpha) = m_\zeta$ for all $\alpha \in I_\zeta$, where φ is from Lemma 7.14.
- (ii) $\{\text{supp}(p_\zeta^\alpha) : \alpha \in I_\zeta\}$ forms a nice Δ -system with kernel S_ζ , moreover $\alpha \in \text{supp}(p_\zeta^\alpha) \setminus S_\zeta$.
- (iii) $\langle x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha \rangle = \langle x_\zeta, y_\zeta, z_\zeta, w_\zeta \rangle$ for $\alpha \in I_\zeta$.

Then $\{x_\zeta^\alpha, y_\zeta^\alpha, z_\zeta^\alpha, w_\zeta^\alpha\} = \{x_\zeta, y_\zeta, z_\zeta, w_\zeta\} \subset S_\zeta \times \omega$.

Find $m \in \omega$ and $I \in [\omega_1]^{\omega_1}$ such that

(iv) $m_\zeta = m$ for all $\zeta \in I$, and so

$$\forall \zeta \in I \forall \alpha \in I_\zeta \varphi(p_\zeta^\alpha) = m.$$

(v) $\{S_\zeta : \zeta \in I\}$ forms a nice Δ -system with kernel S .

Pick $\{\xi, \zeta\} \in [I]^2$. Then pick $\alpha \in I_\zeta$ such that $S_\xi \cup S_\zeta < \text{supp}(p_\zeta^\alpha) \setminus S_\zeta$. So

$$S < (S_\xi \cup S_\zeta) \setminus S < \text{supp}(p_\zeta^\alpha) \setminus S_\zeta.$$

Now pick $\beta \in I_\xi$ such that $\text{supp}(p_\zeta^\alpha) < \text{supp}(p_\xi^\beta) \setminus S_\xi$. So

$$S < (S_\xi \cup S_\zeta) \setminus S < \text{supp}(p_\zeta^\alpha) \setminus S_\zeta < \text{supp}(p_\xi^\beta) \setminus S_\xi.$$

Thus $\text{supp}(p_\zeta^\alpha) \cap \text{supp}(p_\xi^\beta) = S$, $\alpha \in \text{supp}(p_\zeta^\alpha) \setminus S_\zeta$ and $\beta \in \text{supp}(p_\xi^\beta) \setminus S_\xi$.

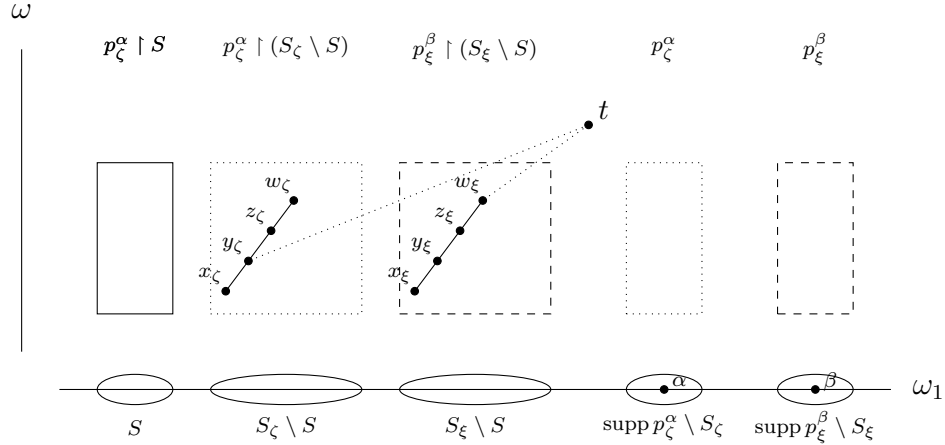
Since $\varphi(p_\zeta^\alpha) = \varphi(p_\xi^\beta)$, the conditions $\varphi(p_\zeta^\alpha)$ and $\varphi(p_\xi^\beta)$ are twins, and

$$q = p_\zeta^\alpha \oplus p_\xi^\beta$$

is a common extension. Pick $t \in (\alpha \times \omega) \setminus (A_\zeta^\alpha \cup A_\xi^\beta)$ with $y_\zeta \triangleleft t$ and $w_\xi \triangleleft t$.

Define r as follows:

$$r = \langle A^q, \preceq_q \cup \langle y_\zeta, t \rangle \cup \langle w_\xi, t \rangle, I^q, \{T_\alpha^q \cup \{t\}, T_\beta^q \cup \{t\}, T^\gamma : \gamma \in I^q \setminus \{\alpha, \beta\}\}, f^q, g^q \rangle.$$



To check $r \in P$ we will use the following observation:

$$r \restriction (\text{supp}(p_\zeta^\alpha) \cup \{t\}) = p_\zeta^\alpha \uplus_{y_\zeta^\alpha} \{t\}_\alpha \quad (7.1)$$

and

$$r \restriction (\text{supp}(p_\xi^\beta) \cup \{t\}) = p_\xi^\beta \uplus_{w_\xi^\beta} \{t\}_\beta. \quad (7.2)$$

Now let us check (P1)–(P5).

(P1) is trivial for r .

(P2). Let $\gamma \in I^q$. If $\gamma \neq \alpha, \beta$, then $T_\gamma^q = T_\gamma^p$, so we are done.

Moreover, $T_\alpha^r = T_\alpha^q \cup \{t\}$, $t \in \alpha \times \omega$, and $\langle T_\alpha^r, \preceq \rangle$ is a tree by (7.1) and (7.2).

The same argument works for T_β^r .

(P3) is trivial.

(P4)(a). Assume that $\gamma \in I^r$, $x, y \in T_\gamma^r$ with $U^r(x) \cap U^r(y) \neq \emptyset$. Since $U^r(t) = \{t\}$ we can assume $x, y \in A^q$.

Assume that $\gamma \in I_\zeta^\alpha$. Then $T_\gamma^q \subset A_\zeta^\alpha$, and so $x, y \in A_\zeta^\alpha$. Thus $t \in U^r(x) \cap U^r(y)$ implies $y_\zeta^\alpha \in U^r(x) \cap U^r(y)$. So $U^q(x) \cap U^q(y) \neq \emptyset$, which yields that x and y are \preceq^q comparable because $q \in P$.

Similar argument works when $\gamma \in I_{\xi}^{\beta}$.

(P4)(b). Assume that $\{\alpha', \beta'\} \in \text{dom}(f^r) = \text{dom}(f^q) = \text{dom}(p_\zeta^\alpha) \cup \text{dom}(p_\xi^\beta)$. We can assume that $\{\alpha', \beta'\} \in \text{dom}(p_\xi^\beta)$.

Write $n = f^r(\{\alpha', \beta'\})$.

(i) Assume on the contrary that there are $a \in T_{\alpha'}^r(n)$ and $b \in T_{\beta'}^r(n)$ with $U^r(a) \cap U^r(b) \neq \emptyset$.

First assume that $\{a, b\} \in [A^q]^2$. Since $q \in P$, we have $U^q(a) \cap U^q(b) = \emptyset$. So $t \in U^r(a) \cap U^q(b)$ should hold.

If $c \in A_\zeta^\alpha$, then $t \in U(c)$ implies $y_\zeta \in U(c)$ by 7.1. Similarly, if $c \in A_\xi^\beta$, then $t \in U(c)$ implies $w_\xi \in U(c)$ by 7.2.

Since $U^q(a) \cap U^q(b) = \emptyset$, we can assume that $a \in A_\zeta^\alpha \setminus A_\xi^\beta$ and $b \in A_\xi^\beta \setminus A_\zeta^\alpha$.

But then $\alpha' \in \text{supp}(p_\zeta^\alpha) \setminus S$ and $\beta' \in \text{supp}(p_\xi^\beta) \setminus S$, so $f^r(\alpha', \beta')$ is undefined. Contradiction.

So we can assume that e.g. $t = a$ and $b \in A^q$. Assume first that $b \in A^{p_\zeta^\alpha}$. Then $\alpha' = \alpha$ and $y_\zeta \in A_\zeta^\alpha$ by (7.1). Thus $y_\zeta \in T_{\alpha'}^{p_\zeta^\alpha}(< n) \cap U^{p_\zeta^\alpha}(b)$, and so $T_{\alpha'}^{p_\zeta^\alpha}(< n) \cap U[T_{\beta'}^{p_\zeta^\alpha}(n)] \neq \emptyset$, so (P4)(b) fails for p_ζ^α .

If $b \in A_\xi^\beta$, then we can use similar arguments using (7.2) instead of (7.1).

(ii) Assume on the contrary that there are $a \in T_{\alpha'}^r(n)$ and $b \in T_{\beta'}^r(< n) \cap U^r(a)$.

Clearly $a \neq t$. If $b \neq t$, then $a \in T_{\alpha'}^q(n)$ and $b \in T_{\beta'}^q(< n) \cap U^q(a)$ which contradicts $q \in P$.

Assume that $b = t$. If $b \in A^{p_\zeta^\alpha}$, then (7.1) implies $\beta' = \alpha$ and $y_\zeta \in U^q(a) \cap T^q(< n)$. Thus $y_\zeta \in T_{\beta'}^q(< n) \cap U^q(a)$, which contradicts $q \in P$.

If $b \in A^{p_\xi^\beta}$, then we can use similar arguments using (7.2) instead of (7.1).

(P5). Let $\langle x, \gamma \rangle \in \text{dom}(g^r)$ and $y \in T_\gamma^r(g(x, \gamma))$

Since $U^r(t) = \{t\}$, we can assume that $x, y \neq t$.

So $x, y \in A^q$. If $U^q(y) \subset U^q(x)$, then $x \preceq^q y$ and so $U^r(y) \subset U^r(x)$.

Assume on the contrary that $U^q(x) \cap U^q(y) = \emptyset$, but $t \in U^r(x) \cap U^r(y)$.

We can assume that $\langle x, \gamma \rangle \in g^{p_\xi^\alpha}$. Thus $x \in A_\xi^\alpha$ and $\gamma \in I_\xi^\alpha$.

However $T_\gamma^q \subset A_\xi^\alpha$, so $y \in A_\xi^\alpha$.

Since $x, y \in A_\xi^\alpha$ and $\gamma \in I_\xi^\alpha$, $t \in U^r(x) \cap U^r(y)$ implies $y_\xi \in U^{p_\xi^\alpha}(x) \cap U^{p_\xi^\alpha}(y)$ by (7.1), which contradicts $U^q(x) \cap U^q(y) = \emptyset$.

So we proved $r \in P$.

Next we show that $r \leq p_\xi^\alpha, p_\xi^\beta$. (O1)–(O4) are trivial. To check (O5), assume on the contrary that $U^{p_\xi^\alpha}(a) \cap U^{p_\xi^\alpha}(b) = \emptyset$, but $U^r \cap U^r(b) \neq \emptyset$.

Then $t \in U^r(a) \cap U^r(b)$, and so $y_\xi^\alpha \in U^{p_\xi^\alpha}(a) \cap U^{p_\xi^\alpha}(b)$ by (7.1), which is a contradiction.

Finally, it is also straightforward that

$$r \Vdash (\text{G5})(i) \text{--}(ii) \text{ holds for } \alpha, \beta, \zeta, \xi, \text{ and } t. \quad (7.3)$$

So we proved the theorem. \square

8. OPEN PROBLEMS

In this section, we present a list of open problems which could be of further interest and are closely connected to our results.

Problem 8.1. *Is every linearly ordered space base resolvable?*

Problem 8.2. *Is every T_3 (hereditarily) separable space base resolvable?*

Problem 8.3. *Is every paracompact space base resolvable?*

Note that under PFA, every T_3 hereditarily separable space is Lindelöf hence base resolvable by Corollary 3.6. Also, we conjecture that our forcing construction can be modified to produce a separable non base resolvable space.

Problem 8.4. *Is every power of \mathbb{R} base resolvable? Is it true that base resolvability is preserved by products?*

We know that every π -base is the union of two disjoint π -bases by Proposition 2.3. However:

Problem 8.5. *Does every base contain a disjoint base and π -base?*

Bases closed to finite unions are resolvable by Corollary 4.7 which raises to following question:

Problem 8.6. *Is it true that every base which is closed to finite intersections is base resolvable?*

It would be interesting to look into the following:

Problem 8.7. *Is every self filling family \mathcal{F} of closed (Borel) sets of ω^ω resolvable?*

Concerning negligible subsets we ask the following:

Problem 8.8. *Is there a base \mathbb{B} for some space X such that every $\mathcal{U} \in [\mathbb{B}]^{|\mathbb{B}|}$ contains a neighborhood base at some point?*

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